

UNIQUENESS OF GLOBAL STRONG SOLUTIONS TO THE 2D NAVIER-STOKES-VLASOV EQUATIONS

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ABSTRACT. The uniqueness of global strong solutions to the Navier-Stokes-Vlasov equations in two spatial dimensions is proved by a priori estimates and semigroup analysis.

1. INTRODUCTION

The goal of this paper is to establish the uniqueness of global strong solutions for two dimensional Navier-Stokes-Vlasov equations:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P - \mu \Delta \mathbf{u} &= - \int_{\mathbb{R}^2} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}, \\ \operatorname{div} \mathbf{u} &= 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_x f + \operatorname{div}_{\mathbf{v}}((\mathbf{u} - \mathbf{v})f) &= 0 \end{aligned} \tag{1.1}$$

in $(0, T) \times \mathbb{R}^2 \times \mathbb{R}^2$, with the following initial data

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad f(x, \mathbf{v}, 0) = f_0(x, \mathbf{v}), \tag{1.2}$$

where \mathbf{u} is the velocity of the fluid, p is the pressure, μ is the kinematic viscosity of the fluid. Without loss of generality, we take $\mu = 1$ throughout the paper. A distribution $f(t, x, \mathbf{v})$ depends on the time $t \in [0, T]$, the physical position $x \in \mathbb{R}^2$ and the velocity of particle $\mathbf{v} \in \mathbb{R}^2$. The number of particles enclosed at $t \geq 0$ and location $x \in \mathbb{R}^2$ in the volume element $d\mathbf{v}$ is given by $f(t, x, \mathbf{v}) \, d\mathbf{v}$. We refer the readers to [3, 6, 7, 8, 9] for more physical background and discussion of Navier-Stokes-Vlasov equations and related problems.

There have been many mathematical analysis of Navier-Stokes-Vlasov equations and the related problems. The global existence results for Stokes-Vlasov system in a bounded domain was established in [6]. The existence theorem for weak solutions has been extended in [2], where the author did not neglect the convection term and considered the Navier-Stokes-Vlasov equations within a periodic domain. The weak solutions of Navier-Stokes-Vlasov-Poisson system with corresponding boundary value problem was obtained in [1]. The global existence of smooth solutions with small data for Navier-Stokes-Vlasov-Fokker-Planck equations was obtained in [5]. More Recently, the existence theorem for global weak solutions with large data for Navier-Stokes-Vlasov equations in a bounded domain was established in [10].

However, there is no existence theory available for the Navier-Stokes-Vlasov equations with initial data in the whole space. Compared to [2, 10], the new difficulty is the loss of

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compactness of $\int_{\mathbb{R}^2} f d\mathbf{v}$ and $\int_{\mathbb{R}^2} \mathbf{v}f d\mathbf{v}$ in the whole space. So the ideas in [2, 10] do not work here. Motivated by the work of [4], we shall show the uniqueness of global strong solutions in two dimensional spaces by a priori estimates and semigroup analysis. Due to the regularity, we cannot extend our method to three dimensional space.

In what follows, we denote

$$m_k f = \int_{\mathbb{R}^2} |\mathbf{v}|^k f d\mathbf{v}, \quad \text{and} \quad M_k f = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathbf{v}|^k f d\mathbf{v} dx,$$

$$\rho = \int_{\mathbb{R}^2} f d\mathbf{v}, \quad j = \int_{\mathbb{R}^2} \mathbf{v}f d\mathbf{v}.$$

It is easy to see that

$$M_k f = \int_{\mathbb{R}^2} m_k f dx.$$

Here we state the following lemma due to [6]:

Lemma 1.1. *Suppose that (\mathbf{u}, f) be a smooth solution to (1.1)-(1.2). If $f_0 \in L^p$ for any $p > 1$, we have*

$$\|f(t, x; \mathbf{v})\|_{L^p} \leq e^{2T} \|f_0\|_{L^p}, \quad \text{for any } t \geq 0;$$

and if $|\mathbf{v}|^k f_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, then we have

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}|^k f d\mathbf{v} dx \leq C(2, T) \left(\left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} |\mathbf{v}|^k f_0 d\mathbf{v} dx \right)^{\frac{1}{2+k}} + (\|f_0\|_{L^\infty} + 1) \|\mathbf{u}\|_{L^r(0, T; L^{2+k})} \right)^{2+k}.$$

Our main result reads as follows

Theorem 1.1. *If $\mathbf{u}_0 \in W^{1,2}(\mathbb{R}^2)$ is a divergence-free vector, $f_0 \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $M_6 f_0 < \infty$, then there exists a unique strong solution (\mathbf{u}, f) to (1.1)-(1.2) for any $T > 0$.*

The strong solution to system (1.1)-(1.2) is defined as follows:

Definition 1.1. A pair (\mathbf{u}, f) is called a strong solution to the system (1.1)-(1.2) if

- $\mathbf{u} \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2))$;
- $f(t, x, \mathbf{v}) \geq 0$, for any $(t, x, \mathbf{v}) \in (0, T) \times \mathbb{R}^2 \times \mathbb{R}^2$;
- $f \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$;
- $f|\mathbf{v}|^2 \in L^\infty(0, T; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$.

2. A PRIORI ESTIMATES

The aim of this section is to obtain some a priori estimates, and we start deducing the energy inequality. Multiplying by \mathbf{u} the both sides of the first equation in (1.1), integrate over \mathbb{R}^2 and by parts, we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} \frac{1}{2} |\mathbf{u}|^2 dx + \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\mathbf{u} - \mathbf{v}) \mathbf{u} d\mathbf{v} dx. \quad (2.1)$$

Multiplying by $(1 + \frac{1}{2}|\mathbf{v}|^2)$ the both sides of the third equation in (1.1) and integrate over \mathbb{R}^2 and by parts, one obtains that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(1 + |\mathbf{v}|^2) d\mathbf{v} dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f |\mathbf{u} - \mathbf{v}|^2 d\mathbf{v} dx \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f (\mathbf{u} - \mathbf{v}) \mathbf{u} d\mathbf{v} dx. \end{aligned} \quad (2.2)$$

Using (2.1)-(2.2), one obtains

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}^2} |\mathbf{u}|^2 dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(1 + |\mathbf{v}|^2) d\mathbf{v} dx \right) + 2 \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx \\ &+ 2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} f |\mathbf{u} - \mathbf{v}|^2 d\mathbf{v} dx \leq 0. \end{aligned} \quad (2.3)$$

Taking the curl of the first equation in (1.1) we obtain

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega - \Delta \omega = \nabla^T \cdot (-\mathbf{u} \rho + j), \quad (2.4)$$

where $\omega = \text{curl} \mathbf{u}$. Multiplying by ω the both sides of (2.4) and integrating we obtain, after integration by parts, we have

$$\frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \leq C (\|\mathbf{u}\|_{L^4}^4 + \|\rho\|_{L^4}^4 + \|j\|_{L^2}^2). \quad (2.5)$$

On the other hand, by (2.3), we have $\mathbf{u} \in L^2(0, T; W^{1,2}(\mathbb{R}^2))$, which implies

$$\mathbf{u} \in L^2(0, T; L^p(\mathbb{R}^2)), \quad \text{for any } p \geq 1.$$

Thus we have $M_6 f < \infty$ by lemma 1.1. Applying Lemma 1 in [2] in two-dimensional space, we can control $\|\rho\|_{L^4}$ and $\|j\|_{L^2}$ by $M_6 f$. Thus, we obtain that

$$\sup_{0 \leq t \leq T} \|\omega\|_{L^2}^2 + \int_0^T \|\nabla \omega\|_{L^2}^2 dt \leq C(T). \quad (2.6)$$

Thus we proved

Proposition 2.1. *Let (\mathbf{u}, f) be a solution of (1.1)-(1.2) on $[0, T]$, with $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^2)$ and $f_0 \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $M_6 f_0 \leq C < \infty$, then we have the following regularity:*

$$\begin{aligned} & \mathbf{u} \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2)); \\ & f \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)). \end{aligned}$$

3. LOCAL AND GLOBAL STRONG SOLUTION

Proposition 3.1. *Let $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^2)$ be a divergence-free vector, $f_0 \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, $M_6 f_0 \leq C < \infty$, then there exists a time $T_0 > 0$ depending on the initial data and a unique strong solution*

$$\begin{aligned} & \mathbf{u} \in L^\infty(0, T_0, \mathbb{P}W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T_0, \mathbb{P}W^{2,2}(\mathbb{R}^2)); \\ & f \in L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \end{aligned}$$

of (1.1) with the initial data (\mathbf{u}_0, f_0) , where \mathbb{P} is the Leray-Hodge projector on divergence-free vector.

Proof. We define $\|(\mathbf{u}, f)\|_B = \|\mathbf{u}\|_X + \|f\|_Y$, where

$$X = L^\infty(0, T_0, \mathbb{P}W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T_0, \mathbb{P}W^{2,2}(\mathbb{R}^2)),$$

$$\|\mathbf{u}\|_X = \|\mathbf{u}\|_{L^\infty(0, T_0, \mathbb{P}W^{1,2}(\mathbb{R}^2))} + \|\mathbf{u}\|_{L^2(0, T_0, \mathbb{P}W^{2,2}(\mathbb{R}^2))};$$

and

$$Y = L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2),$$

$$\|f\|_Y = \|f\|_{L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)}.$$

Clearly the spaces X, Y are Banach spaces, and thus B is Banach space.

We let $U = (\mathbf{u}, f)$ in the Banach space B , define the operator $U = T(U)$ in B , where $T(U) = (\bar{\mathbf{u}}, \bar{f})$. Here $\bar{\mathbf{u}}, \bar{f}$ are given by the following ones

$$\begin{aligned} \bar{\mathbf{u}} &= e^{t\Delta} \mathbf{u}_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\mathbf{u} \cdot \nabla \mathbf{u}) ds + \int_0^t e^{(t-s)\Delta} \mathbb{P}(-\rho \mathbf{u} + j) ds, \\ \bar{f} &= N(\mathbf{u}, \mathbf{v}) \quad \text{where} \quad \partial_t \bar{f} + \mathbf{v} \cdot \nabla \bar{f} + \operatorname{div}_{\mathbf{v}}((\mathbf{u} - \mathbf{v}) \bar{f}) = 0, \quad \bar{f}(x, \mathbf{v}, 0) = f_0(x, \mathbf{v}). \end{aligned} \quad (3.1)$$

We denote $Q(\mathbf{u}, w) = \int_0^t e^{(t-s)\Delta} \mathbb{P}(\mathbf{u} \cdot \nabla w)$, which solves

$$\partial_t Q - \Delta Q = \mathbb{P}(\mathbf{u} \cdot \nabla w), \quad Q(x, 0) = 0.$$

It is easy to obtain the following energy inequality,

$$\frac{d}{dt} \|\Delta Q\|_{L^2}^2 + \|\nabla \Delta Q\|_{L^2}^2 \leq \|\nabla(\mathbf{u} \cdot \nabla w)\|_{L^2} \|\nabla \Delta Q\|_{L^2},$$

then Ladyzhenskaya inequality for term involving $\nabla \mathbf{u} \cdot \nabla w$ and the interpolation inequality for the term involving $\mathbf{u} \cdot \nabla(\nabla w)$ to yield

$$\sup_{0 \leq t \leq T} \|\Delta Q\|_{L^2}^2 + \int_0^{T_0} \|\nabla \Delta Q\|_{L^2}^2 dt \leq CT_0 \|\mathbf{u}\|_X^2 \|w\|_X^2.$$

We denote $L := \int_0^t e^{(t-s)\Delta} \mathbb{P}(-\rho \mathbf{u} + j) ds$, which solves

$$\partial_t L - \Delta L = \mathbb{P}(-\rho \mathbf{u} + j), \quad L(x, 0) = 0.$$

We multiply ΔL the both sides of above equation, and use integration by parts,

$$\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + 2 \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq \int_0^{T_0} \|\Delta L\|_{L^2}^2 ds + \int_0^{T_0} \|\rho \mathbf{u}\|_{L^2}^2 ds + \int_0^{T_0} \|j\|_{L^2}^2 ds,$$

thus we have

$$\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq \int_0^{T_0} (\|\rho\|_{L^4}^2 + \|\mathbf{u}\|_{L^4}^2 + \|j\|_{L^2}^2) ds.$$

Applying Lemma 1 of [2] and Lemma 1.1, we can control $\|\rho\|_{L^4}$ as follows

$$\|\rho\|_{L^4} \leq M_6 f \leq C(M_6 f_0 + \|\mathbf{u}\|_X)^8.$$

Similarly, we can control the term $\|j\|_{L^2}$. Thus, we have the following estimate:

$$\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq C(1 + \|\mathbf{u}\|_X)^8.$$

We estimate $N(\mathbf{u}, \mathbf{v})$ as follows

$$\|N(\mathbf{u}, \mathbf{v})\|_{L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))} \leq e^{2T_0} \|f_0\|_Y.$$

Following the same argument of [2], we let $g = e^{-2t}f$ satisfy the following transport equation

$$\partial_t g + \mathbf{v} \cdot \nabla g + (\mathbf{u} - \mathbf{v}) \cdot \nabla_{\mathbf{v}} g = 0.$$

The above equation can be written by characteristics method as follows,

$$\begin{aligned} \frac{dx}{dt} &= \mathbf{v}(t), \\ \frac{d\mathbf{v}}{dt} &= \mathbf{u}(t, x(t)) - \mathbf{v}(t), \end{aligned}$$

with the initial data

$$x(0) = x \quad \text{and} \quad \mathbf{v}(0) = \mathbf{v},$$

and set $\chi(t, x, \mathbf{v}) = (x(t), \mathbf{v}(t))$ for any (t, x, \mathbf{v}) . Thus, we got the following solution

$$f(t, x, \mathbf{v}) = e^{2t} f_0(\chi(t, x, \mathbf{v})), \quad \text{for any } (t, x, \mathbf{v}).$$

Thus we have

$$\|f_1 - f_2\|_Y \leq C \|\chi_1 - \chi_2\|_Y. \quad (3.2)$$

By the definition of $\chi = (x, \mathbf{v})$, we have the following estimate

$$\begin{aligned} & \|(\chi_1 - \chi_2)(t)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \\ & \leq C \left(\int_0^t \|(\mathbf{u}_1 - \mathbf{u}_2)(s)\|_{L^\infty(\mathbb{R}^2)} ds + \int_0^t (1 + \|\mathbf{u}(s)\|_X) \|(\chi_1 - \chi_2)(s)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} ds \right) \\ & \leq C \left(\int_0^t \|(\mathbf{u}_1 - \mathbf{u}_2)(s)\|_{L^\infty(\mathbb{R}^2)} ds + \int_0^t \|(\chi_1 - \chi_2)(s)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} ds \right). \end{aligned}$$

Thus, for any t , apply Gronwall's inequality, to have

$$\|\chi_1 - \chi_2\|_Y \leq C\varepsilon \|\mathbf{u}_1 - \mathbf{u}_2\|_X.$$

This, together with (3.2), implies that

$$\|f_1 - f_2\|_Y \leq C\varepsilon \|\mathbf{u}_1 - \mathbf{u}_2\|_X.$$

Thus

$$\|N(\mathbf{u}_1, \mathbf{v}) - N(\mathbf{u}_2, \mathbf{v})\|_Y \leq C\varepsilon \|\mathbf{u}_1 - \mathbf{u}_2\|_X. \quad (3.3)$$

We establish the iteration $U^{n+1} = T(U^n)$. It is easy to see that the sequences U^n is bounded in B and convergence exponentially fast if we choose ε small enough. If there exist A , D and ε such that

$$\|\mathbf{u}^n\|_X \leq A, \quad \|f^n\|_Y \leq D,$$

then, by introduction and (3.1), we have

$$\|\mathbf{u}^{n+1}\|_X \leq A_0 + \varepsilon A^2 + \varepsilon C(1 + A)^8, \quad \|f^{n+1}\|_Y \leq e^{2T_0} D_0.$$

we can choose ε small enough, such that $\varepsilon(1 + A)^8 + \varepsilon A^2 + A_0 \leq A$, and choose D such that $e^{2T_0} D_0 \leq D$. Thus, we conclude that the sequence is bounded in B , then we can obtain the exponential convergence of \mathbf{u}^n in X , f^n in Y .

□

By Proposition 3.1, there exists a unique strong solution on a short time interval $[0, T_0]$. For any given $T_0 > 0$, there exists a constant $K > 0$ such that

$$e^{2T_0} \|f\|_Y \leq K,$$

which implies $\|f(T_0, x, \mathbf{v})\|_Y \leq K$. This, together with the energy inequality and particular a priori estimates in Section 2, and apply Proposition 3.1, the unique strong solution can be extended to $[T_0, T_0 + T^*]$. One repeat many times, such a unique strong solution can be extended to the whole time interval $[0, T]$ for any $T > 0$.

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